Bulk Solution of Ginzburg-Landau Equations for Type II Superconductors: Upper Critical Field Region

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A solution is described of the Ginzburg–Landau equations of the Abrikosov type for a homogeneous type II superconductor just below the upper critical field. This solution is characterized by magnetic field maxima corresponding to an equilateral triangular lattice and a value $\beta=1.16$ of Abrikosov's parameter. Abrikosov's square-lattice solution with $\beta=1.18$ has a higher free energy and is unstable with respect to the triangular lattice solution.

An approximate solution of the Ginzburg–Landau equations appropriate for homogeneous type II superconductors in magnetic fields just below the upper critical field H_{c2} has been found by Abrikosov[1]. For this solution the field distribution and the "superconducting electron" distribution are microscopically periodic transverse to the uniform applied field H_0, and have square symmetry. Abrikosov conjectures that this solution is the solution of lowest free energy. We have discovered a solution which has lower free energy, points of zero-order parameter which correspond to a triangular lattice, and points of maximum-order parameter on a honeycomb lattice. This is the lowest energy solution of a one-parameter continuum of solutions with respect to which Abrikosov's solution is unstable. These solutions are described below in Abrikosov's notation.

We begin with Abrikosov's general approximate solution

$$\Psi(x,y)=\sum_{n=-\infty}^{\infty} C_n \exp(inky) \exp[-\tfrac{1}{2}\kappa^2(x-nk/\kappa^2)^2] \quad (1)$$

with the periodicity condition $C_{n+N}=C_n$. The C_n and k are to be adjusted to minimize the free energy. We take $N=2$, which is the case next simplest to the $N=1$ case treated by Abrikosov, and require that the order parameter $\omega=|\Psi|^2$, proportional to the "superconducting electron" concentration, be invariant under a centering translation with respect to the rectangular cell. This cell has sides $L_x=2k/\kappa^2$, $L_y=2\pi/k$, and area $4\pi/\kappa^2$ independent of the ratio $R=L_x/L_y=k^2/\pi\kappa^2$ of the sides. The centering translational symmetry condition imposes on ω the translational symmetry of an equilateral triangular lattice when R is suitably adjusted. This condition, $\omega(x+\frac{1}{2}L_x, y+\frac{1}{2}L_y)=\omega(x,y)$, leads directly to the relation $C_1=\pm iC_0$. The geometric nature of the corresponding solutions $\Psi_\pm$ is conveniently characterized by their points of equal ω value (equivalent points). The points $(x+[m+\frac{1}{2}q]L_x, y+[n+\frac{1}{2}q]L_y)$ with m, n, and q integers are translationally equivalent to (x,y). The zeros, which are particularly helpful in elucidating the geometry, correspond to normal filaments where $H=H_0-(1/2\kappa)\omega$ is maximum. They occur at points translationally equivalent to $(\frac{1}{4}L_x, \frac{1}{4}L_y)$ for ω_+ and to $(\frac{1}{4}L_x,\frac{3}{4}L_y)$ for ω_-. In addition, $(\frac{1}{2}L_x,0)$ and $(0,0)$ are equivalent for $\omega_\pm$; also, $\omega_+(0,0)=\omega_-(0,0)$.

The Gibbs free-energy density corresponding to (1)

$$G=F_1-2H_0B=\tfrac{1}{2}-H_0^2-(\kappa-H_0)^2/(2\kappa^2-1)\beta, \quad (2)$$

readily derived from Abrikosov's expressions for the free-energy density and magnetic induction, decreases monotonically with decreasing values of the parameter $\beta=\langle\omega^2\rangle_{\rm av}/\langle\omega\rangle_{\rm av}^2$, for fixed values of κ and H_0 (where the average is a spatial one). For $N=2$,

$$\beta=\frac{2}{(2\pi)^{\frac{1}{2}}}\frac{k}{\kappa}\frac{(|C_0|^4+|C_1|^4)f_0^2+4|C_0|^2|C_1|^2f_0f_1+2\,{\rm Re}(C_0^{*2}C_1^2)f_1^2}{(|C_0|^2+|C_1|^2)^2}, \quad (3)$$

where the

$$f_q=f_q(R)\equiv\sum_{m=-\infty}^{\infty} e^{-(\pi/2)R(2m+q)^2}$$

are related to theta functions. The minimum of (3) with respect to C_1/C_0 occurs when the centering translation symmetry is present and is given by

$$\beta(R)=(R/2)^{\frac{1}{2}}(f_0^2+2f_0f_1-f_1^2)=\beta(1/R). \quad (4)$$

The relation $\beta(R)=\beta(1/R)$, which might be expected on symmetry grounds, can be verified by eliminating f_1 in (4) with the aid of $f_0(R)+f_1(R)=f_0(R/4)$, and using the Poisson sum identity $f_0(R)=(1/2R)^{\frac{1}{2}}f_0(1/4R)$. The $\beta(R)$ of (4) is plotted in Fig. 1. Its minimum value $\beta=1.1596$ occurs for $R=3^{\frac{1}{2}}$ with $\omega_\pm$, shown in Fig. 2, having hexagonal symmetry ($p6mm$) and zeros on a triangular lattice with nearest-neighbor distance

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[1] A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957)].

$(4\pi/3^{\frac{1}{2}}\kappa^2)^{\frac{1}{2}}=L_y$. The Fourier series for ω with $R=3^{\frac{1}{2}}$ is

$$\omega_\pm(x,y)=|C_0|^2 3^{-\frac{1}{4}} \sum_{m,n=-\infty}^{\infty} (\pm)^n(-)^{mn}$$

$$\times\exp[-\pi 3^{-\frac{1}{2}}(m^2+mn+n^2)]\exp[2\pi i(mu+nv)], \quad (5)$$

where $u=2x/3^{\frac{1}{2}}L_y$, $v=(3^{-\frac{1}{2}}x+y)/L_y$.

The maximum value $\beta=1.1803$, which occurs at $R=1$, corresponds to $\omega_\pm$ having square symmetry ($p4mm$) and zeros on a square lattice with nearest neighbor distance $L_1=(2\pi/\kappa^2)^{\frac{1}{2}}$. This solution is equivalent to the square lattice solution of Abrikosov[2] rotated by 45° and translated; by comparing the Fourier series for ω in the two cases, one finds

$$\omega_{N=1}(x,y)=\omega_-\left(\frac{x+y}{2^{\frac{1}{2}}}+\tfrac{1}{4}L, \frac{-x+y}{2^{\frac{1}{2}}}+\tfrac{1}{4}L\right)$$

$$=|C_0|^2 \sum_{m,n=-\infty}^{\infty} (-)^{mn}\exp\left[-\frac{\pi}{2}(m^2+n^2)\right]$$

$$\times\exp[2\pi i(mx+ny)/L_1], \quad (6)$$

where $L=L_x=L_y=2^{\frac{1}{2}}L_1=(4\pi/\kappa^2)^{\frac{1}{2}}$. The symmetry properties of ω are more readily determined from the Fourier series than from (1). Also, one observes that the origin is not in general a point of maximum ω.

From the above results one sees that, in the field range below H_{c2} where an approximate solution of the type of (1) holds, Abrikosov's solution with square symmetry is unstable, and is continuously connected with the triangular lattice solution by a pure shear deformation of the normal filament lattice structure.

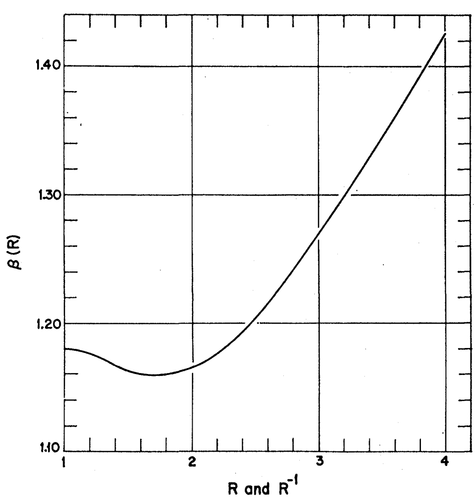

FIG. 1. Parameter β for the $N=2$ minimum free-energy solution of the Ginzburg-Landau equations just below the upper critical field H_{c2} versus the ratio $R=L_x/L_y$ of the rectangular unit-cell dimensions. β approaches $(R/2)^{\frac{1}{2}}$ for large R.

[2] See Abrikosov's first figure for a contour plot analogous to Fig. 2.

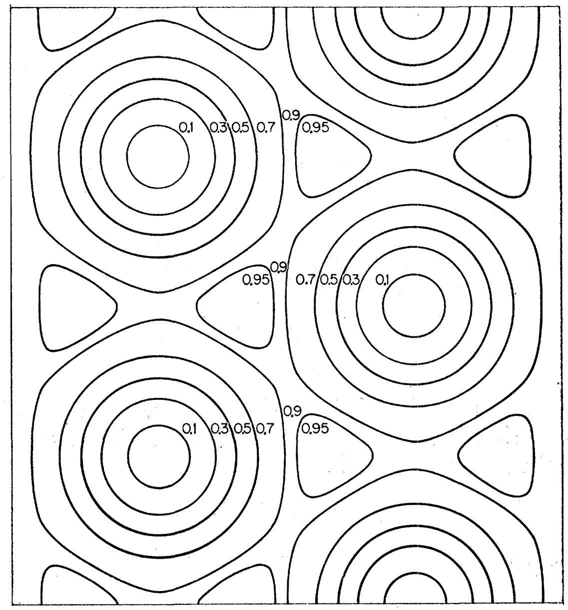

FIG. 2. Contour diagram of $\omega=|\Psi|^2$ for the $N=2$ minimum free-energy solution of the Ginzburg-Landau equations just below the upper critical field H_{c2} for the ratio $R=L_x/L_y=3^{\frac{1}{2}}$ of the rectangular unit-cell dimensions. The maximum of ω is here normalized to unity. The origin in Eq. (1) is at a point equivalent to the saddle point closest to the lower left corner of the diagram for ω_+ and closest to the upper left corner for ω_-.

Using an approximation of a different nature, Abrikosov[1] finds for $\kappa\gg 1$ that a triangular lattice solution is stable just above the lower critical field H_{c1}, but that a first-order phase transformation to a square lattice solution occurs at $H_1'=H_{c1}+0.0394/\kappa$. On this basis our result implies that with increasing applied field an additional phase transformation back to the triangular lattice structure must occur. It may be, however, that a more accurate calculation near the lower critical field would show that no first-order transition occurs, in which case the triangular lattice solution would hold throughout the mixed state.

The value of β affects the slope of the magnetization curve, which is linear[1] near H_{c2},

$$B-H_0=\beta^{-1}(2\kappa^2-1)^{-1}(H_0-H_{c2}); \quad (7)$$

but it appears that the triangular and square lattice solutions would be hard to distinguish by measuring this slope, which differs by only 2% in the two cases. A diffraction experiment might succeed in distinguishing the two structures. In a real superconductor, however, periodic arrays of the filaments may be considerably altered due to crystal imperfections.

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